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Instability of the tunneling destruction effect in a quasi-periodically driven two-level system

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Abstract. Here we consider the dynamics of a two-level system under an external time-dependent field. We show that in the case of a bichromatic field the dynamical localization effect is strongly sensitive with respect to the commensurability of the driving frequencies.

PACS. 03.65.-w Quantum mechanics – 73.40.Gk Tunneling

Driven two-level systems have been the subject of great interest since the pioneering works by Rabi who solved the problem of a two-level spin system in a circularly polarized magnetic field [1]. At present, they appear in many fields, from theoretical physics to practical optics (see [2–10] and references therein). Furthermore, the concept of driven two-level system is the corner-stone of quantum computing [11,12].

One of the main address concerns dynamical localization. In a seminal paper Grossmann and co-workers [13] pointed out that the beating motion in a two-level system can be controlled, and even suppressed, by means of a tailored external monochromatic driving field. Applications resulting from this discovery are, among others, the confinement of electrons in quantum structures [14], the control of molecular reactions [15,16] and the control of electron transfer [17].

Recently, the trapping mechanism by means of two laser fields has been explored and, on the basis of numerical investigations, a chaotic dependence from the external field's parameters is suggested [18,19]. In this paper, we give the theoretical explanation of this chaotic behavior and we show that the influence, on the dynamical localization effect, of a change of the field's frequencies cannot be reduced beyond a certain limit.

A generic equation describing the dynamics of a two-level driven system can be written as

$$\mathrm{i}\dot{\phi} = H_1\phi, \quad H_1 = \epsilon\sigma_1 + f(t)\sigma_3, \; \phi(0) = \phi^0, \tag{1}$$

where $\epsilon > 0$ is the beating frequency, $\dot{\phi}$ denotes the derivative of ϕ with respect to the time t,

$$\phi(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix},$$

$f(t)$ is the driving force that depends on time and $\sigma_{1,3}$ are the two Pauli's matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The actual semiclassical parameter is the beating frequency ϵ.

For our purposes [20] will be useful to write the original equation (1) in a different form by means of the transformation

$$\psi = \mathrm{e}^{\mathrm{i}\alpha\sigma_3}\phi$$

where

$$\alpha(t) = \int_0^t f(\xi)\mathrm{d}\xi. \tag{2}$$

Then equation (1) takes the form

$$\mathrm{i}\dot{\psi} = H_2\psi, \quad H_2 = \epsilon \mathrm{e}^{\mathrm{i}\alpha\sigma_3}\sigma_1\mathrm{e}^{-\mathrm{i}\alpha\sigma_3}, \tag{3}$$

with the same initial condition $\psi(0) = \phi^0$.

When the driving field is absent, that is $f(t) \equiv 0$, then equation (3) has a periodic solution and the imbalance function, defined as

$$z(t) = |\psi_1(t)|^2 - |\psi_2(t)|^2 \equiv |\phi_1(t)|^2 - |\phi_2(t)|^2, \tag{4}$$

is a periodic function, which assumes both positive and negative values. The beating period is $T = \pi/\epsilon$.

It has been found that for a monochromatic driving force the wavefunction ϕ is, for certain values of the field's parameters, nearly "frozen" in its initial configuration [13]. Indeed, let

$$f(t) = \frac{1}{2}\eta\sin(\omega t) \tag{5}$$

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where η and ω are, respectively, the amplitude and the frequency of the external monochromatic field. By means of the average theorem [21], in the limit of small beating frequency, that is $\epsilon \ll \omega$, we can approximate the solution of equation (3) by the solution of the average system given by

$$\mathrm{i}\dot{\psi} = \hat{H}_2\psi,\ \hat{H}_2 = \epsilon J_0(\eta/\omega)\sigma_1 \tag{6}$$

where $J_0(x)$ is the zeroth Bessel function. Hence, the unperturbed imbalance function $z(t)$ is approximated by means of the imbalance function $\hat{z}$ related to the average equation (6) for any time of the order $1/\epsilon$. That is, for any $\delta > 0$ there exists $\epsilon_0 > 0$ such that for any ϵ, $0 < \epsilon < \epsilon_0$, then

$$|z(t) - \hat{z}(t)| < \delta,\quad \forall t \in [0,T],\ T = \frac{\pi}{\epsilon}\cdot$$

From this fact and since (6) has the same form of equation (1) with $f(t) \equiv 0$ and ϵ replaced by $\epsilon J_0(\eta/\omega)$ then it follows that $z(t)$ is, up to a small correction, a periodic function, with beating period now given by $\pi/\epsilon J_0(\eta/\omega)$, that assumes both positive and negative values.

In particular, when the external field's parameters η and ω are such that $J_0(\eta/\omega) = 0$ then the beating motion disappears and we have dynamical localization. By means of a continuity argument we have that the dynamical localization effect is still observed for any value of the field's parameters such that η/ω is close enough to a zero of the zero-th Bessel function $J_0(x)$. That is we have that:

Proposition 1. *When the parameters of the driving field (5) are such that $\eta/\omega \approx x_0$, where $J_0(x_0) = 0$, then we have dynamical localization.*

This result could suggest us to plan an experiment where we can "freeze" a two-level system, at least for any time of the order of the beating period T, by means of an external monochromatic driving field for suitable values of the amplitude and of the frequency. In fact, a small change of these parameters does not actually affect the result of the experiment because it is not necessary that the ratio η/ω is exactly equal to x_0, but we only need that this ratio is close enough to x_0.

This favorable situation does not hold in the cases of external monochromatic driving fields, with modulation of the amplitude, or bichromatic driving fields. In fact, in such a case we'll show that any change, small at will, of the field's frequencies actually affects in a chaotic way the behavior of the solution of equation (3).

Let us consider, for the sake of argument, the monochromatic field with modulation of the amplitude given by

$$f(t) = \eta \sin(\Omega t)\sin(\omega t),\ \epsilon \ll \Omega < \omega.$$

From standard trigonometric formulas it follows that this case is equivalent to the case of an external bichromatic field with same amplitude η and different frequencies $\omega_1 = \omega - \Omega$ and $\omega_2 = \omega + \Omega$:

$$f(t) = \frac{1}{2}\eta\left[\cos(\omega_1 t) - \cos(\omega_2 t)\right]. \tag{7}$$

Now, let us recall the following theoretical result [20]. Let $\alpha(t)$ be defined as in (2) and let

$$I(t) = \frac{1}{t}\int_0^t \mathrm{e}^{2\mathrm{i}\alpha(\xi)}\mathrm{d}\xi.$$

If the limit

$$\hat{I} = \lim_{t\to\infty} I(t)$$

exists and it is zero then we have dynamical localization; that is the solution ψ of equation (3) is "frozen" in to its initial value and $z(t) \sim z(0)$ for any $t \in [0,T]$. In order to apply this result we compute, by means of the Bessel functions, the explicit expression of the limit $\hat{I}$ when $f(t)$ is given by (7). We have that [20]

$$\hat{I} = J_0(\eta/\omega_1)J_0(\eta/\omega_2) + r, \tag{8}$$

where the remainder term r is given by

$$r = \begin{cases} 0 & \text{if } \frac{\omega_2}{\omega_1} \in R - Q \\ \sum_{\ell=-\infty,\ell\neq 0}^{+\infty} J_{n\ell}(\eta/\omega_1)J_{m\ell}(\eta/\omega_2) & \text{if } \frac{\omega_2}{\omega_1} = \frac{n}{m} \in Q \end{cases}$$

where n and m are two integer numbers which have no common divisor. In particular, r is exactly zero when the two frequencies ω_1 and ω_2 are incommensurate.

Let us assume, for a moment, that the frequency ω of the field is exactly three times the frequency Ω of the amplitude modulation, that is $\omega_2 = 2\omega_1$, $n = 2$ and $m = 1$. Then (8) takes the form

$$\hat{I} = \sum_{\ell=-\infty}^{+\infty} J_{2\ell}(\mu)J_\ell(\mu/2),\quad \mu = \frac{\eta}{\omega_1},$$

and equation $\hat{I} = 0$ has a solution for $\mu = \mu_0$, where $\mu_0 = 3.593\ldots$ Hence, we can state that:

Proposition 2. *When the parameters of the driving field (7) are such $\eta/\omega_1 \approx \mu_0$ and ω_2 is exactly the twice of ω_1 then we have dynamical localization.*

Despite the appearance, we have that Proposition 2 is much more weak than Proposition 1. In fact, the continuity argument, applied to Proposition 1, does not fully apply in this second case. In order to show this fact let $\eta/\omega_1 = \mu_0$ and let ω_2 be almost, but not exactly, the twice of ω_1, for instance $\omega_2 = 2.01\omega_1$. In such a case we have that

$$\frac{\omega_2}{\omega_1} = \frac{201}{100} \in Q,\ n = 201,\ m = 100$$

and (this result is exact if ω_1 and ω_2 are incommensurate),

$$\hat{I} = \sum_{\ell=-\infty}^{+\infty} J_{201\ell}(\eta/\omega_1)J_{100\ell}(\eta/2.01\omega_1)$$
$$\approx J_0(\mu_0)J_0(\mu_0/2.01) = -0.135749 \neq 0$$

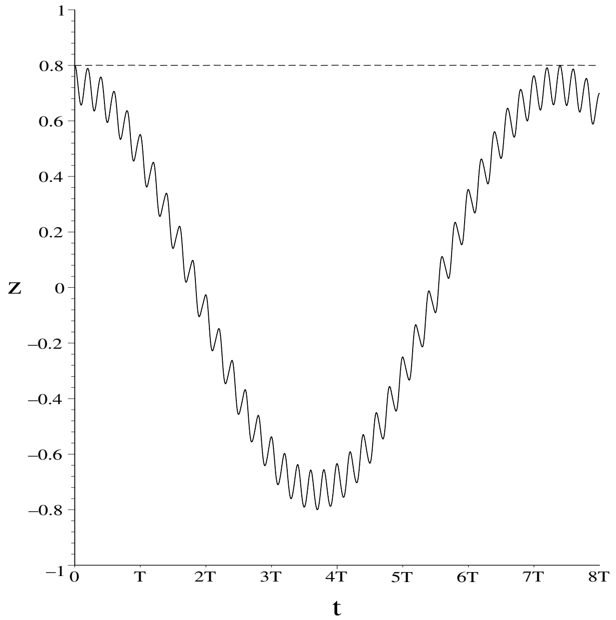

Fig. 1. In this figure we plot the graphs of the imbalance function $z(t) = |\psi_1(t)|^2 - |\psi_2(t)|^2$. Broken line represents the case of ω_2 exactly equal to the twice of ω_1 and $\eta/\omega_1 = \mu_0 \approx 3.593\ldots$, in such a case we have dynamical localization. If ω_2 is almost, but not exactly, equal to the twice of ω_1, *e.g.* $\omega_2 = 2.01\omega_1$, then, for the same value of the ratio $\eta/\omega_1 = \mu_0$, we don't have dynamical localization (bold line). T is the beating period.

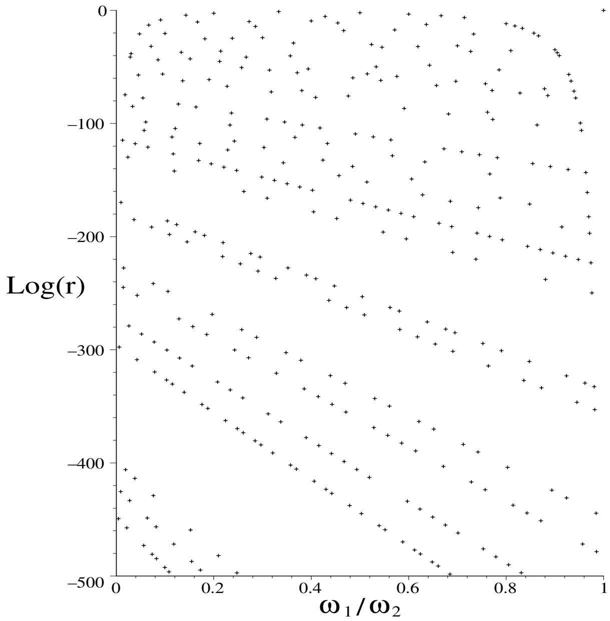

Fig. 2. In this figure we plot the logarithmic graph of the function r with respect to the ratio ω_1/ω_2 where we have fixed, for argument sake, $\eta/\omega_1 = 2$. From this picture a chaotic behavior appears.

since $|J_n(z)| \leq |z/2|^n/n!$ [22]. Therefore, we don't have now the dynamical localization effect since $\hat{I} \neq 0$. As a result we can conclude that Proposition 2 does not hold when ω_2 is not exactly the twice of ω_1.

The numerical evidence of this fact could be seen by introducing the imbalance function $z(t)$, defined in (4), and the relative phase

$$\theta(t) = \arg[\psi_1(t)] - \arg[\psi_2(t)].$$

By means of a simple computation, from equation (3) it follows that these two functions have to satisfy the following system of ordinary differential equations:

$$\begin{cases} \dot{z} = -2\epsilon\sqrt{1-z^2}\sin\theta \\ \dot{\theta} = 2\epsilon\cos\theta\frac{z}{\sqrt{1-z^2}} - 2f(t) \end{cases}$$

We compute, now, the numerical solution of these equations for, *e.g.*, the initial conditions $z(0) = 0.8$ and $\theta(0) = 0$ in the two cases:

(a) $\epsilon = 0.01$, $\omega_1 = 1$, $\omega_2 = 2$ and $\eta/\omega_1 = \mu_0$, where we expect to have dynamical localization;
(b) $\epsilon = 0.01$, $\omega_1 = 1$, $\omega_2 = 2.01$ and $\eta/\omega_1 = \mu_0$, where we expect to don't have dynamical localization.

Indeed, in full agreement with the above conclusion, we see in Figure 1 that the effect of dynamical localization is very sensitive with respect to the field's parameters.

It is important to underline that this situation appears, at least theoretically, for any couple of values ω_1 and ω_2, not only in the case $\omega_2 = 2\omega_1$ (even if in such a case the numerical evidence is more easy to obtain).

The very basic reason of the phenomenon discussed above is explained by means of the chaotic dependence of the function $r(\omega_1, \omega_2, \eta)$ from the field's frequencies. In order to be more precise we observe that, for any given value ω_1, ω_2 and η of the field's parameter, we have:

$$\min_{\chi>0} \max_{\gamma\in(1-\chi,1+\chi)} |\hat{r} - r(\omega_1, \gamma\omega_2, \eta)| = |\hat{r}|.$$

where $\hat{r} = r(\omega_1, \omega_2, \eta)$. This property is a direct consequence of the fact that $r = 0$ if the two frequencies ω_1 and ω_2 are incommensurate and from the fact that the Bessel functions $J_n(x)$ decrease very fast with respect to n.

Hence, for any η and ω_1 fixed, it follows that $\hat{I}$ is a discontinuous function on the set

$$S = S_{\omega_1,\eta} = \{\omega_2\colon r(\omega_1, \omega_2, \eta) \neq 0\}$$

and this set S is a dense set on the real axis (see Fig. 2). Therefore, we can conclude that the influence of a small change of the field's frequencies on the dynamical localization effect cannot be reduced beyond a certain limit by improving the resolution.

In conclusion, in this paper we have explored the dynamical localization effect for two-level systems under the effect of a bichromatic external field. We have proved that this effect appears only when the limit $\hat{I}$ is zero and we have also proved, in the specific case (7), that $\hat{I}$ depends on the driving frequencies in a very discontinuous way. As a result, the theoretical explanation of the chaotic behavior predicted by Wilkens and Rzazewski [19] is given.

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